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AN EXACT DETERMINATION OF THE EFFECTIVE MODULI OF ELASTICITY OF MICRO-INHOMOGENEOUS MEDIA†

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An exact solution is obtained of the problem of determining all the macroscopic elastic constants of an anisotropic textured polycrystalline system comprising grains with cubic symmetry of the elastic properties and a discrete distribution of those properties in two equivalent orientations, differing by a rotation about the fourth-order axis through an angle of $\pi/4$. © 1999 Elsevier Science Ltd. All rights reserved.

Exact solutions have been obtained of the problem of finding the effective moduli of elasticity of micro-inhomogeneous materials only for media comprising orthotropic layers which are arbitrarily distributed over the thickness [1, 2] and for composites of isotropic phases with identical shear moduli [3]. In the latter case the singularities of the spatial distribution of the phases play no part and the material remains macroscopically isotropic. Further analysis shows that an exact solution can also be obtained in the special case of an anisotropic micro-inhomogeneous medium consisting of grains of cubic symmetry with two ideal orientations, which change from one to the other on rotation through an angle of $\pi/4$ about the common axis of symmetry.

The elastic behaviour of micro-inhomogeneous materials is described by the generalized Hooke's law, which gives the relation between the strain and stress tensors ε and σ

$$\boldsymbol{\varepsilon} = \mathbf{s}\boldsymbol{\sigma}, \quad \boldsymbol{\sigma} = \mathbf{c}\boldsymbol{\varepsilon} \tag{1}$$

where $\mathbf{s}$ and $\mathbf{c}$ are the tensors of the compliances and moduli of elasticity at any point of the medium.

Since the elements of inhomogeneity are so small and are randomly distributed, the micro-inhomogeneous medium can be regarded as macroscopically uniform and characterized by a set of effective compliances $\mathbf{s}^*$ and effective moduli of elasticity $\mathbf{c}^*$, which relate the characteristics of the strain and stress fields averaged over the volume of the system

$$\langle\boldsymbol{\varepsilon}\rangle = \mathbf{s}^*\langle\boldsymbol{\sigma}\rangle, \quad \langle\boldsymbol{\sigma}\rangle = \mathbf{c}^*\langle\boldsymbol{\varepsilon}\rangle \tag{2}$$

Averaging in (1) and comparing with (2), we obtain

$$\mathbf{s}^*\langle\boldsymbol{\sigma}\rangle = \langle\mathbf{s}\boldsymbol{\sigma}\rangle$$

After identity transformations we have

$$\mathbf{s}^*\langle\boldsymbol{\sigma}\rangle = \langle\mathbf{s}\rangle\langle\boldsymbol{\sigma}\rangle + \langle(\mathbf{s}-\langle\mathbf{s}\rangle)\boldsymbol{\sigma}\rangle \tag{3}$$

We take as the micro-inhomogeneous medium a polycrystal with cubic symmetry in which the grains have only two ideal orientations of equal relative volume concentration and an angle of disorientation about the common fourth-order axis of symmetry of $\pi/4$. The axis of symmetry of the system coincides with the x_3 axis of the laboratory system of coordinates. Then the matrices of the compliances of grains of the first and second orientations in the laboratory system have the form

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$$s^1 = \begin{Vmatrix} \mathbf{S}_1^1 & \mathbf{O} \\ \mathbf{O} & \mathbf{S}_2^1 \end{Vmatrix}, \quad s^2 = \begin{Vmatrix} \mathbf{S}_1^2 & \mathbf{O} \\ \mathbf{O} & \mathbf{S}_2^2 \end{Vmatrix}$$

$$\mathbf{S}_1^1 = \begin{Vmatrix} s_{11} & s_{12} & s_{12} \\ s_{12} & s_{11} & s_{12} \\ s_{12} & s_{12} & s_{11} \end{Vmatrix}, \quad \mathbf{S}_1^2 = \begin{Vmatrix} s_{11} - \zeta/2 & s_{12} + \zeta/2 & s_{12} \\ s_{12} + \zeta/2 & s_{11} - \zeta/2 & s_{12} \\ s_{12} & s_{12} & s_{11} \end{Vmatrix}$$

$$\mathbf{S}_2^1 = \mathrm{diag}\,\{s_{44}, s_{44}, s_{44}\}, \quad \mathbf{S}_2^2 = \mathrm{diag}\,\{s_{44}, s_{44}, s_{44} + 2\zeta\}$$

$$\zeta = s_{11} - s_{12} - s_{44}/2$$

($\mathbf{O}$ is the 3×3 zero matrix).

The matrix of mean values of the compliances takes the form

$$\langle s \rangle = \begin{Vmatrix} \mathbf{S}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{S}_2 \end{Vmatrix}$$

$$\mathbf{S}_1 = \begin{Vmatrix} s_{11} - \zeta/4 & s_{12} + \zeta/4 & s_{12} \\ s_{12} + \zeta/4 & s_{11} - \zeta/4 & s_{12} \\ s_{12} & s_{12} & s_{11} \end{Vmatrix}, \quad \mathbf{S}_2 = \mathrm{diag}\{s_{44}, s_{44}, s_{44} + \zeta\}$$

This polycrystalline system has an axis of symmetry of order eight, which is also the axis of elastic symmetry of infinite order [4], and so the material is transversely isotropic [5]. From the condition for transverse isotropicity of the system it follows that

$$s^*_{66} = 2\,(s^*_{11} - s^*_{12}) \tag{4}$$

Considering the different stressed states of the system, namely, the macroscopic stressed states corresponding to extension along the x_1 and x_3 axes, simple shear in the x_2, x_3 planes as well as uniform compression, from Eq. (3) and the form of the matrices s^1, s^2 and $\langle s \rangle$ we obtain

$$s^*_{13} = s_{12}, \quad s^*_{33} = s_{11}, \quad s^*_{44} = s_{44}, \quad s^*_{11} + s^*_{12} = s_{11} + s_{12} \tag{5}$$

From (4) and (5) it follows that all the effective compliances can be found merely by finding the coefficient s^*_{66}. We do this by considering two stressed states of simple shear

$$\langle \sigma \rangle = \begin{Vmatrix} 0 & \langle \sigma_{12} \rangle & 0 \\ \langle \sigma_{12} \rangle & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \text{and}\, \langle \sigma' \rangle = \begin{Vmatrix} -\langle \sigma_{12} \rangle & 0 & 0 \\ 0 & \langle \sigma_{12} \rangle & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

($\langle \sigma' \rangle$ is obtained by rotating the stress field $\langle \sigma \rangle$ through an angle of $\pi/4$ about the x_3 axis of the laboratory system).

By virtue of the symmetry of the system, a rotation of the stress field is equivalent to the interchange of the mean stresses in grains of different orientations, that is

$$\langle \sigma' \rangle_1 = \langle \sigma \rangle_2, \quad \langle \sigma' \rangle_2 = \langle \sigma \rangle_1 \tag{6}$$

where $\langle \sigma \rangle_i$ and $\langle \sigma' \rangle_i$ are the tensors of the mean stress over the volume occupied by grains of the ith orientation.

There are similar relations for the mean strains

$$\langle \varepsilon' \rangle_1 = \langle \varepsilon \rangle_2, \quad \langle \varepsilon' \rangle_2 = \langle \varepsilon \rangle_1 \tag{7}$$

The relation between σ' and ε' at an arbitrary point of the system is also given by the generalized Hooke's law

$$\varepsilon' = s\sigma', \quad \sigma' = c\varepsilon' \tag{8}$$

The tensors of the compliances and moduli of elasticity at an arbitrary point of the system can be written in the form

$$\mathbf{s} = \lambda(\mathbf{s}^1 - \mathbf{s}^2) + \mathbf{s}^2, \quad \mathbf{c} = \lambda(\mathbf{c}^1 - \mathbf{c}^2) + \mathbf{c}^2 \tag{9}$$

where $\mathbf{c}_1$, $\mathbf{c}_2$ are the tensors of the moduli of elasticity of the corresponding orientations, and λ is a random indicator function, which is equal to one when the point belongs to grains of the first orientation and zero otherwise.

Substituting expressions (9) for the compliance tensor and moduli of elasticity tensor into relation (8) and averaging using (6) and (7), we obtain

$$\langle\sigma'\rangle = (\mathbf{c}^1\langle\varepsilon\rangle_2 + \mathbf{c}^2\langle\varepsilon\rangle_1)/2, \quad \langle\varepsilon'\rangle = (\mathbf{s}^1\langle\sigma\rangle_2 + \mathbf{s}^2\langle\sigma\rangle_1)/2$$

It follows from the invariance of the rotation that $\langle\sigma'\rangle$ and $\langle\varepsilon'\rangle$ relate the same tensor $\mathbf{s}^*$ of effective compliances. Then

$$\mathbf{s}^1\langle\sigma\rangle_2 + \mathbf{s}^2\langle\sigma\rangle_1 = \mathbf{s}^*(\mathbf{c}^1\langle\varepsilon\rangle_2 + \mathbf{c}^2\langle\varepsilon\rangle_1) \tag{10}$$

where $\langle\varepsilon\rangle_1 = \mathbf{s}_1\langle\sigma\rangle_1$, $\langle\varepsilon\rangle_2 = \mathbf{s}_2\langle\sigma\rangle_2$ and Eq. (10) can be rewritten (in coordinate form) as

$$s^1_{66}\langle\sigma_{12}\rangle_2 + s^2_{66}\langle\sigma_{12}\rangle_1 = s^*_{66}\left(c^1_{66}s^2_{66}\langle\sigma_{12}\rangle_2 + c^2_{66}s^1_{66}\langle\sigma_{12}\rangle_1\right) \tag{11}$$

The mean shear stresses over the volumes occupied by the separate orientations occurring here are found by averaging in (1) over the whole volume of the polycrystalline specimen using (9). We obtain

$$\langle\sigma_{12}\rangle_l = r_{12}\langle\sigma_{12}\rangle, \quad r_{ik} = 2(s^*_{66} - s^k_{66})/(s^i_{66} - s^k_{66})$$

A similar expression is obtained for $\langle\sigma_{12}\rangle_2$ by permuting the indices of the various orientations of the grains.

Substituting the resulting quantities into (11) we obtain

$$r_{21}s^1_{66} + r_{12}s^2_{66} = s^*_{66}\left(r_{21}s^2_{66}/s^1_{66} + r_{12}s^1_{66}/s^2_{66}\right)$$

Solving the resulting quadratic equation, we have

$$c^*_{66} = [c_{44}(c_{11} - c_{12})/2]^{1/2}$$

that is, the macroscopic shear modulus in a plane perpendicular to the isotropic plane and in a direction lying in the isotropic plane is found as the geometric mean of the shear moduli in the plane of the face and the diagonal plane of the elementary cubic crystal lattice.

The remaining macroscopic compliances are given by Eqs (4) and (5). Finally, for the moduli of elasticity, we obtain

$$c^*_{11} = (c_{11} + c_{12})/2 + c^*_{66}, \quad c^*_{12} = (c_{11} + c_{12})/2 - c^*_{66}$$

$$c^*_{33} = c_{11}, \quad c^*_{13} = c_{12}, \quad c^*_{44} = c_{44}$$

This solution holds whatever the shape of the grains, provided the elastic symmetry of the material is preserved. In particular, it holds for a two-dimensional polycrystalline system with a "honeycomb" structure in which the grains are infinite hexagonal prisms.

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